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A Fluctuation-Dissipation Relation and Its Applications to Thermodynamic Fluctuations in Superconducting Cylinders

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Modifying slightly Kubo's formulation of perturbation theory to take care of the fact that generalized susceptibility may not be zero at infinite frequency, we establish, in the classical limit, a general relation between covariance and generalized susceptibility. The relation is then applied to evaluate the covariances of the magnetic flux, currents, and magnetization of a superconducting cylinder. Expressions for the spectra of magnetic flux and magnetization are also obtained.

KEY WORDS: Linear response; generalized susceptibility; covariance; spectrum; relaxation; superconducting cylinder; magnetic flux; current; magnetization.

1. INTRODUCTION

One of the developments in the theory of irreversible processes is the expression in closed form of generalized susceptibilities in terms of the correlation functions.[(1)] After calculating (e. g., by the help of the Greens function technique) the correlation functions microscopically, we arrive at an expression for the generalized susceptibility (or complex admittance) to a mechanical perturbation such as a magnetic or electric field. Modern physics is full of such instances. Sometimes, it is more convenient, however, to evaluate the generalized susceptibilities and then express the correlations in terms of these susceptibilities. The case of a superconducting cylinder is such an example. Due to the existence of boundaries, it is rather mathematically involved to evaluate the time correlation functions microscopically. With some simplifying assumptions, it is, however, straightforward to derive explicit expressions for

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the generalized susceptibilities. Then, variances and covariances can be obtained with the aid of the relations that connect them to generalized susceptibilities.

The fluctuation-dissipation theorem[2] provides a well-known contact between the generalized susceptibility of a quantity and its spectrum. Between generalized susceptibility and the cross-spectrum of two quantities, an analogously simple relationship cannot always be established. We need to know in more detail about the time reversibility of the quantities involved. In Section 2, however, by modifying slightly Kubo's formalism to take care of the fact that $\alpha_{BA}(\infty) \neq 0$, we establish, in the classical limit, a general relation between the generalized susceptibility, $\alpha_{BA}(\omega)$, and the cross-spectrum of A and B. When $\omega = 0$, a particularly simple relation between the generalized susceptibility and the covariance of A and B can be derived. This relation is utilized in Section 3 to evaluate covariances of the magnetic flux, currents, and magnetization of a superconducting cylinder. In Section 4, we employ the fluctuation-dissipation theorem to evaluate the spectra of magnetic flux and magnetization. The main cause of fluctuation lies in the random scattering of excitations (normal electrons) by lattices and impurities in the superconductors. Although the average of the current carried by the normal electrons in the absence of a time-dependent field is zero, this normal component of current fluctuates about its average value. Such fluctuations, in turn, give rise to fluctuations in the magnetic flux and in the component of the current carried by the Cooper pairs.

Vant-Hull *et al.*[3] have measured the fluctuation of magnetic flux in a tin rod. The variation of the standard deviation of magnetic flux (rms value of the magnetic noise) with temperature is shown in Fig. 1. The solid curve is plotted according to the theoretical expression (30a) in Section 3. According to this expression, the thermal, magnetic noise does not disappear when the metal becomes superconducting. With the dimensions of the tin rod used in the experiment (10 cm $\times$ 0.47 cm diameter), we expect the standard deviation to drop more than an order of magnitude from its normal-state value as the order of $(T_c - T)/T_c$ becomes greater than 10^{-5}. While there is no contradiction between our theory and the experiment, more accurate data are needed to decide whether our theory is an adequate one to describe the magnetic fluctuations in superconducting cylinders. We hope that the present work can stimulate more efforts in this direction. As pointed out by Vant-Hull *et al.*,[3] the existence of thermal magnetic noise has serious implications in the design of magnetic shielding and for magnetometry in the submicrogauss regime; better understanding about the thermal magnetic fluctuations is therefore important.

2. CORRELATION AND GENERALIZED SUSCEPTIBILITY

Kubo has introduced a very convenient formalism connecting time correlation function and generalized susceptibility $\alpha_{BA}(\omega)$. In the case $\lim_{\omega\to\infty} \alpha_{BA}(\omega) \neq 0$, his formalism, however, needs to be slightly modified. We keep to linear approximation. An operator for a physical quantity $\mathscr{B}$ in the presence of an external force $F(t)$ can then be written, in general, in the form[2]

[2] For example, the current density in an infinite medium in the presence of an external field is

$$\mathbf{j}(\mathbf{r}, t) = \tilde{\mathbf{j}}(\mathbf{r}, t) + (e^2/mc)\Psi(\mathbf{r}, t)^+\Psi(\mathbf{r}, t)\mathbf{A}(\mathbf{r}, t)$$

where $\tilde{\mathbf{j}} = (e/im)[\Psi^+\nabla\Psi - (\nabla\Psi^+)\Psi]$. In this case, $X = (e^2/mc)\Psi(\mathbf{r}, t)^+\Psi(\mathbf{r}, t)$, and $\hbar$ is set to be 1.

$$B = \tilde{B} + \chi F(t) \tag{1}$$

where $\tilde{B}$ and χ are operators that do not involve $F(t)$ explicitly. Suppose the perturbation energy can be written as $\mathscr{H}'(t) = -AF(t)$. The linear response is observed through the change $\Delta B(t)$, which can be expressed as[1]

$$\Delta B(t) = \int_{-\infty}^{t} \phi_{BA}(t - t')\, F(t')\, dt'$$

where the aftereffect function $\phi_{BA}(t - t')$ differs from Kubo's,

$$\phi_{BA}(t) = \tilde{\phi}_{BA}(t) + 2\chi\delta(t) \tag{2}$$

$\tilde{\phi}_{BA}(t)$ is Kubo's aftereffect function,[1] defined by

$$\tilde{\phi}_{BA}(t) \equiv \langle [A, B(t)] \rangle \tag{3}$$

$$[A, B(t)] \begin{cases} \equiv \sum_i \left[\dfrac{\partial A}{\partial q_i} \dfrac{\partial B(t)}{\partial p_i} - \dfrac{\partial A}{\partial q_i} \dfrac{\partial B(t)}{\partial q_i} \right] & \text{(Classical)} \\ \equiv -i[AB(t)]_- \equiv -i[AB(t) - B(t)\, A] & \text{(Quantal)} \end{cases}$$

The generalized susceptibility $\alpha_{BA}(\omega)$ is given by

$$\begin{aligned} \alpha_{BA}(\omega) &= \lim_{\delta \to 0} \int_0^{\infty} \phi_{BA}(t)\, e^{-(i\omega + \delta)t}\, dt \\ &= \tilde{\alpha}_{BA}(\omega) + \langle \chi \rangle \end{aligned} \tag{4}$$

where $\tilde{\alpha}_{BA}(\omega)$ is defined by

$$\tilde{\alpha}_{BA}(\omega) = \lim_{\delta \to 0} \int_0^{\infty} \tilde{\phi}_{BA}(t)\, e^{-(i\omega + \delta)t}\, dt \tag{5}$$

In the case $\tilde{\phi}_{BA}(t)$ possesses no delta-function at $t = 0$ and is almost piecewise continuous (see, e.g., Le Page[4]) for $t > 0$, $\lim_{\omega\to\infty} \tilde{\alpha}_{BA}(\omega) = 0$. In such cases,[3]

$$\lim_{\omega \to \infty} \alpha_{BA}(\omega) = \langle \chi \rangle \tag{6}$$

or

$$\alpha_{BA}(\omega) - \alpha_{BA}(\infty) = \lim_{\delta \to 0} \int_0^{\infty} \tilde{\phi}_{BA}(t)\, e^{-(i\omega + \delta)t}\, dt \tag{7}$$

In the classical limit,[1]

$$\tilde{\phi}_{BA}(t) = -\beta \langle A \dot{B}(t) \rangle$$

[3] For example, for systems with localized electromagnetic response, $\mathbf{j}_\omega(r) = -Q(\omega)\mathbf{A}_\omega(\mathbf{r})$, where $-Q(\omega)$ is the generalized susceptibility, and $\lim_{\omega\to\infty} Q(\omega) = ne^2/mc^2 = -\langle X \rangle$. (See the appendix.)

where $\beta \equiv 1/k_B T$. As $\langle AB(t)\rangle$ is expected to be a continuous function, $\tilde{\phi}_{BA}(t)$ does not possess any delta-function characteristic. We shall keep to classical limit. Equation (7) can be written in the form

$$\alpha_{BA}(\omega) - \alpha_{BA}(\infty) = \lim_{\delta\to 0} \int_{-\infty}^{\infty} \theta(t)\, \tilde{\phi}_{BA}(t)\, e^{-i\omega t - \delta|t|}\, dt$$

where $\theta(t)$ is a step function,

$$\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{8}$$

Putting $\theta(t)$ in an integral form, and changing the order of integration, we have

$$\begin{aligned} \alpha_{BA}(\omega) - \alpha_{BA}(\infty) &= \lim_{\delta\to 0} \frac{-i}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{\omega - \omega' - i\delta} \int_{-\infty}^{\infty} dt[\tilde{\phi}_{BA}(t)\exp(-i\omega' t)] \\ &= -\beta \fint_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\omega'}{\omega - \omega'} S_{BA}(\omega') - \frac{i\beta\omega}{2} S_{BA}(\omega) \end{aligned} \tag{9}$$

where $S_{BA}(\omega)$ is the cross-spectrum of A and B, defined by

$$S_{BA}(\omega) = \int_{-\infty}^{\infty} \langle AB(t)\rangle\, e^{-i\omega t}\, dt \tag{10}$$

The second equality in (9) follows from the relation[5]

$$\int_{-\infty}^{\infty} \tilde{\phi}(t)\, e^{-i\omega t}\, dt = -i\beta\omega \int_{-\infty}^{\infty} \langle AB(t)\rangle\, e^{-i\omega t}\, dt$$

This is the general, but rather involved, relationship between suceptibility and the cross-spectrum of two quantities. In case $S_{BA}(\omega)$ is real, we have[2]

$$S_{BA}(\omega) = -k_B T \times 2[\mathrm{Im}\{\alpha_{BA}(\omega)\}]/\omega \tag{11}$$

We have used the fact that $\mathrm{Im}\{\alpha_{BA}(\omega)\}$ is an odd function of frequency and therefore vanishes at $\omega = \infty$. When $A = B$, $S_{BA}(\omega)$ is always real and we have the familiar fluctation-dissipation theorem. Eqation (11) is in general much simpler than (9), as it does not involve the principal value of an integral of the cross-spectrum. In general, however, S_{BA} is not real[4] and (11) is not true. Nevertheless, some simple and general relationship can be established between generalized susceptibility and covariance. From (9), putting $\omega = 0$, we obtain

$$\alpha_{BA}(0) - \alpha_{BA}(\infty) = \beta \int_{-\infty}^{\infty} d\omega'\, [S_{BA}(\omega')/2\pi]$$

[4] For example, in the case $B = dA/dt$, $S_{BA}(\omega)$ is pure imaginary.

This follows from the fact that $S_{BA}(\omega)$ has no pole at $\omega = 0$. Therefore, we obtain

$$\mathrm{Cov}(A, B) = k_B T[\alpha_{BA}(0) - \alpha_{BA}(\infty)] \tag{12}$$

In the particular case $A = B$, cov(A, B) becomes the variance of A.[6,7] To arrive at 12, we have implicitly assumed $\lim_{t\to\infty}\langle AB(t)\rangle = \langle A\rangle\langle B\rangle$. As pointed out by Kubo,[1] this need not be the case unless the degrees of freedom associated with the observed quantities A and B are much smaller than those of the total system. In Sections 3 and 4, where Eq. (12) is going to be applied, we can assume that a heat reservoir has also been included in the unperturbed system. In this way, we ensure the validity of the relation $\lim_{t\to\infty}\langle AB(t)\rangle = \langle A\rangle\langle B\rangle$.

In order to obtain (12), we only made use of the continuity of $\langle AB(t)\rangle$. However, we can mention in passing that, should $\langle A\dot{B}(t)\rangle$ possess a delta-function at $t = 0$ [i.e., $\langle A\dot{B}(t)\rangle = (2\langle X_1\rangle\delta(t) + f(t)]$, it produces no additional mathematical difficulty. Equation (12) is simply modified to

$$\alpha(0) - \alpha(\omega) = \beta\langle AB(0_+)\rangle - \langle A\rangle\langle B\rangle \tag{13}$$

In this hypothetical case, $\alpha(\infty)$ is equal to $(\langle X_1\rangle + \langle X\rangle)$ instead of $\langle X\rangle$.

3. FLUCTUATIONS IN SUPERCONDUCTING CYLINDER

3.1. Magnetic Flux

To avoid mathematical complications, we limit ourselves to superconductors where the electromagnetic response is localized.[8] Pure Type I superconductors near the critical temperature, pure Type II superconductors, and dirty superconductors fall in this category.[9] In the case of Type II superconductors, the average magnetic field is limited to values much smaller than the lower critical field[10] H_{c1} so as to avoid the spatially inhomogeneous vertex state (or Schubnikov phase). In such a case, the current density and the vector potential are related by[8,11]

$$\mathbf{j}_\omega(\mathbf{r}) = -Q(\omega)\,\mathbf{A}_\omega(\mathbf{r}) \tag{14}$$

where $\mathbf{j}_\omega(\mathbf{r})$ and $\mathbf{A}_\omega(\mathbf{r})$ are respectively the Fourier transform of the response of the current density, $\Delta\mathbf{j}(\mathbf{r}, t)$, and the Fourier transform of the response of the vector potential, $\Delta\mathbf{A}(\mathbf{r}, t)$, with $\Delta\mathbf{A}(\mathbf{r}, t) = [\mathbf{A}(\mathbf{r}, t) - \bar{\mathbf{A}}(\mathbf{r})]$ and $\Delta\mathbf{j}(\mathbf{r}, t) = [\mathbf{j}(\mathbf{r}, t) - \bar{\mathbf{j}}(\mathbf{r})]$. Here, $\mathbf{A}(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ are respectively the average of the vector potential operator and the average of the current density operator in the presence of an external time-dependent magnetic field, $\bar{\mathbf{A}}(\mathbf{r})$ and $\bar{\mathbf{j}}(\mathbf{r})$ are the averages of these operators when there is no time-dependent external magnetic field, and $Q(\omega)$ is a function of ω. In this section, we only need the value of $Q(\omega)$ at $\omega = 0$ and $\omega = \infty$. At $\omega = 0$, for superconductors with localized electromagnetic response, $(4\pi/c)\,Q(\omega)$ is usually expressed as

$$(4\pi/c)\,Q(0) = \lambda_s^{-2} \tag{15}$$

where λ_s is the London penetration depth at temperature T. At $\omega = \infty$, it is shown in the appendix that

$$\lim_{\omega\to\infty} (4\pi/c)\, Q(\omega) = 1/\lambda^2 \tag{16}$$

where $\lambda \equiv \lambda_L(0) = [mc^2/4\pi ne^2]^{1/2}$ and $\lambda_L(0)$ is the London penetration depth at zero temperature, with n the total electron density.

Consider a long cylinder with width d, outer radius b, inner radius a, and length $\mathscr{L}$ ($\mathscr{L} \gg b$). There is no assumption about the values of a, b, and d except that we can assume the magnitude of the order parameter, and hence superconducting electron density, to be spatially constant. This assumption is reasonable as long as $\xi(T)$ is greater than the smaller of $\lambda(T)$ or the width d, where $\xi(T)$ is the coherence length[12] at temperature T. Within the coherence length, we can assume the density to be constant. For Type I superconductors, it is always true that $\xi(T) > \lambda(T)$. For Type II superconductor, we have to satisfy $\xi(T) > d$ [or $\xi(T) > b$ for a solid cylinder].

A homogeneous, external magnetic field is placed parallel to the axis of the cylinder. We assume that the external field consists of two parts, a static field H_s and a time-dependent part $H_0 e^{i\omega t}$. With the axis of symmetry as the z axis, we introduce the cylindrical coordinates (r, θ, z). We use the approximation that the current only flows in the θ direction so that the current, vector potential, and other relevant physical quantities depend only on r. We can then choose a gauge to make the vector potential $\mathbf{A}(\mathbf{r})$ have the form $[0, A(r), 0]$. Remembering that $\mathbf{A}(\mathbf{r}) = [0, A(r), 0]$, and with the aid of the Maxwell equation

$$\nabla \times \nabla \times \mathbf{A}(\mathbf{r}, t) = (4\pi/c)\, \mathbf{j}(\mathbf{r}, t) \tag{17}$$

and the boundary conditions[13]

$$[\nabla \times \mathbf{A}(\mathbf{r}, t)]_{r=b} = H_0 e^{i\omega t} + H_s \tag{18}$$

$$[\nabla \times \mathbf{A}(\mathbf{r}, t)]_{r=a} = (2/a)\, A(a) \tag{19}$$

we obtain

$$A_\omega(r) = q^{-1} D(\omega)^{-1}\, [K_2(qa)\, I_1(qr) + I_2(qa)\, K_1(qr)]\, H_0 \tag{20}$$

where

$$D(\omega) = I_0(qb)\, K_2(qa) - I_2(qa)\, K_0(qb) \tag{21}$$

$$q = [(4\pi/c)\, Q(\omega)]^{1/2} \tag{22}$$

I_ν and K_ν are modified Bessel's functions of order ν. The static[14] value $\bar{A}(r)$ can be obtained from (17)–(19) by putting $H_0 = 0$; i.e.,

$$\begin{aligned}\bar{A}(r) = \{&(1/2\pi r) - (\lambda_s/\pi a^2 D_s)[K_0(b/\lambda_s)\, I_1(r/\lambda_s) + I_0(b/\lambda_s)\, K_1(r/\lambda_s)]\}\, \Phi_l \\ &+ (\lambda_s/D_s)[K_2(a/\lambda_s)\, I_1(r/\lambda_s) + I_2(a/\lambda_s)\, K_1(r/\lambda_s)]\, H_s\end{aligned} \tag{23}$$

where $\Phi_l = lhc/2e$ is the fluxoid (l is an integer),[15] and

$$D_s = I_0(b/\lambda_s)\, K_2(a/\lambda_s) - I_2(a/\lambda_s)\, K_0(b/\lambda_s) \tag{24}$$

The Fourier component of the total magnetic flux response $[\Phi_t(t) - \bar{\Phi}_t]$ of the system is

$$\begin{aligned}\Phi_{t\omega} &= 2\pi b A_\omega(b) \\ &= [2\pi b/qD(\omega)][K_2(qa)\, I_1(qb) + I_2(qa)\, K_1(qb)]\, H_0 \end{aligned} \tag{25}$$

Consider an external field produced by a current $\tilde{I}_0$ in a long solenoid. A long superconducting cylinder is placed inside the solenoid, coaxial to the external field. The force acting on the external current $\tilde{I}_0$ due to the presence of the cylinder is given by $(1/c)(d/dt)(\Phi_t - \Phi_0)$, where Φ_t is the total flux in the cylinder. Therefore, the rate of absorption of energy is given by $(\tilde{I}_0/c)\, d(\Phi_t - \Phi_0)/dt$. The perturbation term[2] in the Hamiltonian is then given by $\tilde{I}_0(\Phi_t - \Phi_0)/c$. The generalized susceptibility $\alpha_{\Phi_t}(\omega)$ is defined as

$$\alpha_{\Phi_t}(\omega) = \Phi_{t\omega}/(\tilde{I}_0/c) \tag{26}$$

Using Eq. (25), and remembering that $H_0 = 4\pi\tilde{I}_0/\mathscr{L}c$, $\alpha_{\Phi_t}(\omega)$ is given by

$$\alpha_{\Phi_t}(\omega) = [8\pi^2 b/\mathscr{L}qD(\omega)][K_2(qa)\, I_1(qb) + I_2(qa)\, K_1(qb)] \tag{27}$$

To evaluate the variance of the flux, we have only to calculate $\alpha_{\Phi_t}(0)$ and $\alpha_{\Phi_t}(\infty)$. At $\omega = 0$, $q = 1/\lambda_s$, and

$$\alpha_{\Phi_t}(0) = [8\pi^2 b\lambda_s/\mathscr{L}D_s][K_2(a/\lambda_s)\, I_1(b/\lambda_s) + I_2(b/\lambda_s)\, K_1(b/\lambda_s)] \tag{28a}$$

where D_s is defined by (24). At $\omega = \infty$, $q = 1/\lambda$, and

$$\alpha_{\Phi_t}(\infty) = (8\pi^2 b\lambda/\mathscr{L}D_\lambda)[K_2(a/\lambda)\, I_1(b/\lambda) + I_2(a/\lambda)\, K_1(b/\lambda)] \tag{28b}$$

where

$$D_\lambda = K_2(a/\lambda)\, I_0(b/\lambda) - I_2(a/\lambda)\, K_0(b/\lambda) \tag{29}$$

We notice that $\alpha_{\Phi_t}(\infty)$ has the same value whether we are dealing with superconducting or normal cylinders. The reason for this is that, when $\hbar\omega \gg \Delta$, the energy gap has little effect in influencing electromagnetic absorption. Therefore, the superconductor behaves like a normal conductor at high frequency. In the case where $\omega\tau_n \gg 1$, the normal metal behaves like a free-electron gas. The same remarks apply to the generalized susceptibilities of the other quantities we consider later.

It is good approximation[2,16] to use classical relations for a quantity satisfying $k_B T > 1/\tau$, where τ is the time for that quantity to relax to its steady-state value. For $T = 3°\text{K}$, this condition is satisfied by $\tau > 3 \times 10^{-12}$. For a very pure metal, the momentum relaxation time τ_n is of order 10^{-10} sec. and the time for the magnetic flux, currents, etc. to relax to their metastable values are usually longer than τ_n. Therefore, for most situations of practical interest, we can use the classical relation Eq. (12). Even for the cases of dirty, or very thin, cylinders, where τ_n is a few orders smaller than 10^{-10} sec, the condition $k_B T > 1/\tau$ can still be kept, under proper considerations. In Section 4, further discussion of relaxation times is given.

Using Eq. (12), it is straightforward to obtain[5,6]

$$\mathrm{Var}(\Phi_t) = (8\pi^2 b/\mathscr{L})\, k_B T\{(\lambda_s/D_s)[K_2(a/\lambda_s)\, I_1(b/\lambda_s) + I_2(a/\lambda_s)\, K_1(b/\lambda_s)] - (\lambda/D_\lambda)[K_2(a/\lambda)\, I_1(b/\lambda) + I_2(a/\lambda)\, K_1(b/\lambda)]\} \tag{30}$$

When $T \geqslant T_c$, the cylinder becomes normal. Therefore, $\lambda_s = \infty$. We have then, from Eq. (30),

$$\mathrm{Var}(\Phi_t) = (8\pi^2 b/\mathscr{L})\, k_B T\{(b/2) - (\lambda/D_\lambda)[K_2(a/\lambda)\, I_1(b/\lambda) + I_2(a/\lambda)\, K_1(b/\lambda)]\}$$

We can see that the variance changes continuously when the metal passes from the superconducting to the normal phase.

For a solid superconducting cylinder, we put $a = 0$ in (30), and the variance of the magnetic flux in the cylinder becomes

$$\mathrm{Var}(\Phi_t) = \frac{8\pi^2 b}{\mathscr{L}}\, k_B T \left[\frac{\lambda_s I_1(b/\lambda_s)}{I_0(b/\lambda_s)} - \frac{\lambda I_1(b/\lambda)}{I_0(b/\lambda)}\right] \tag{30a}$$

[5] When the fluxoid[15] is not zero, the variance $\mathrm{Var}(\Phi_t)$ is a measure of the fluctuation of the magnetic flux Φ_t about its metastable value, not about its true equilibrium value. The time needed for, say, magnetic flux, to relax to its true equilibrium value is equal to the lifetime τ_p of the persistent current. τ_p is so very much longer than the time for the magnetic flux to relax back to its metastable value that, for our purposes, we can assume that the system shall never return to its true equilibrium values. Thus, we can evaluate $\mathrm{Var}(\Phi_t)$ as if it is fluctuating about an equilibrium average.

[6] As (19) implies, spatial homogeneity of the field in the hole of the cylinder, (30) is, strictly speaking, only valid for $a < c\tau$ where τ is the relaxation time of Φ_t and c is the velocity of light.

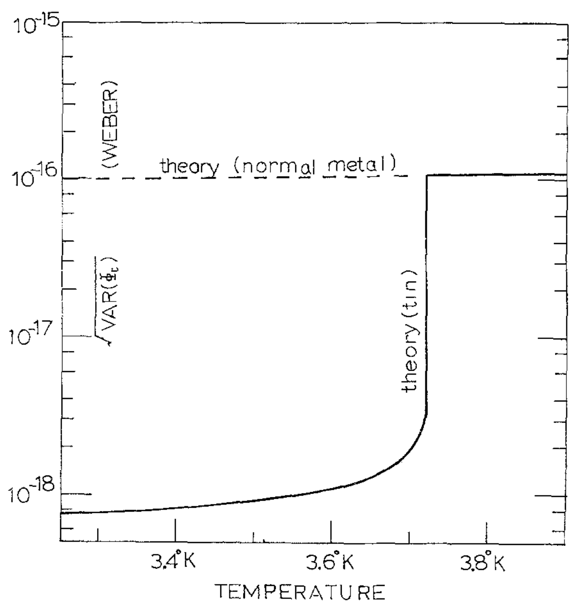

Fig. 1. Thermal magnetic noise in metal rods.

The variation of $\mathrm{Var}(\Phi_t)$ with temperature is plotted in Fig. 1 according to (30a), with $\mathscr{L} = 10$ cm and $b = 0.235$ cm. With these values, the magnetic noise is the same for all the solid metal rods in normal phase. For normal metal rods, $\mathrm{Var}(\Phi_t)$ is represented in Fig. 1 by a dashed line when the temperature is lower than 3.72°K and by a solid line when the temperature is greater than 3.72°K. The solid line represents the magnetic noise in the tin rod, which undergoes a phase transition at 3.72°K. Near the critical temperature, λ/λ_s is approximated by $[2(1 - T/T_c)]^{1/2}$. According to (30a), the thermal fluctuations of magnetic flux do not disappear when the rod becomes superconducting. However, with the given values of $\mathscr{L}$ and b, $\mathrm{Var}(\Phi_t)$ drops more than an order of magnitude when $(1 - T/T_c)$ is greater than 10^{-5}. Vant-Hull *et al.*[3] have measured the thermal magnetic noise of metal rods with the dimensions given above. While there is no contradiction between our theory and the experimental data, more accurate measurements are needed to decide whether or not our theory is adequate to describe the thermal magnetic noise in cylinders.

For a thin cylinder[6,17] such that $a \gg \lambda_s \gg d$ and $ad/2\lambda^2 \gg 1$, the variance of the magnetic flux for a superconducting cylinder becomes, near the critical temperature, where $\lambda_s \gg \lambda$,

$$\mathrm{Var}(\Phi_t) = ck_BTL/[1 + (Rd/2\lambda_s^2)]$$

where L is the self-inductance of a thin cylinder: $L = 4\pi^2R^2/c\mathscr{L}$. In the case of very thin cylinders[11] such that $a \gg \lambda_s$ and $\lambda \gg d$,

$$\mathrm{Var}(\Phi_t) = ck_BTL\frac{Rd/2\lambda_n^2}{(1 + Rd/2\lambda_s^2)(1 + Rd/2\lambda^2)}$$

For the case $a, d \gg \lambda_s$, we have

$$\mathrm{Var}(\Phi_t) = 8\pi^2b(\lambda_s - \lambda)\, k_BT/\mathscr{L}$$

The Fourier component of the total response current $[I(rt) - \bar{I}(r)]$ is given by

$$\begin{aligned} I_\omega &= -(cq^2\mathscr{L}/4\pi)\int_a^b A(r)\,dr \\ &= c\{[2/q^2a^2D(\omega)] - 1\}(\tilde{I}_0/c) \end{aligned} \tag{31}$$

Therefore, the covariance is given by Eq. (12), with $A = \Phi_t$ and $B = I$,

$$\mathrm{Cov}(I, \Phi_t) = (2ck_BT/a^2)[(\lambda_s^2/D_s) - (\lambda^2/D_\lambda)] \tag{32}$$

For very thin cylinders such that $a \gg \lambda_s$ and $d \gg \lambda$, we have

$$\mathrm{Cov}(I, \Phi_t) = ck_BT\frac{Rd/2\lambda_n^2}{(1 + Rd/2\lambda^2)(1 + Rd/2\lambda_s^2)}$$

Denote the magnetic field in the hollow of the cylinder by H_i. For a long cylinder,

H_i can be considered spatially constant. The Fourier transform of $[H_i(t) - \bar{H}]$ is given by

$$H_{i\omega} = (2/a)\, A_\omega(a)$$
$$= [8\pi/a^2 \mathscr{L} q^2 D(\omega)](\tilde{I}_0/c)$$

Therefore, using Eq. (12) with $A = \Phi_t$ and $B = H_i$,

$$\mathrm{Cov}(H_i\,,\, \Phi_t) = (8\pi/\mathscr{L} a^2)[(\lambda_s^2/D_s) - (\lambda^2/D_\lambda)]\; k_B T \tag{33}$$

This expression, in fact, can be obtained from Eq. (32) by aid of the relation

$$H_{i\omega} = H_0 + (4\pi/\mathscr{L} c)\, I_\omega$$

When the cylinder is in the normal phase, the covariance becomes

$$\mathrm{Cov}(H_i\,,\, \Phi_t)_n = (8\pi/\mathscr{L} a^2)[(a^2/2) - (\lambda^2/D_\lambda)]\; k_B T$$

The variation of $\mathrm{Cov}(H_i\,,\, \Phi_t)/\mathrm{Cov}(H_i\,,\, \Phi_t)_n$ with temperature is illustrated in Fig. 2 for cases where $b/\lambda = 100$, $a/b = 0.1$, 0.5, and 0.9. It is of no special significance that this particular ratio of b/λ is chosen. It is chosen simply to be specific.

For thin cylinders, $\mathrm{Cov}(\Phi_i\,,\, \Phi_t) = \mathrm{Var}(\Phi_t) = \mathrm{Var}(\Phi_i)$ where $\Phi_i = \pi a^2 H_i$.

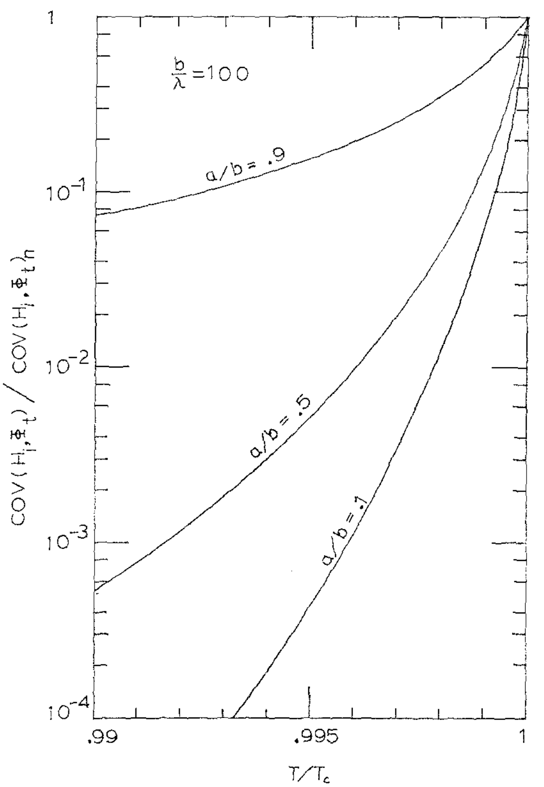

Fig. 2. Variation of the covariance of H_i and Φ_t with temperature in hollow, superconducting cylinders.

3.2. Magnetization

The perturbation term in the Hamiltonian due to the external field can be written as

$$\mathscr{H}'(t) = -\int_{\text{vol}} m(r)\, H_0(t)\, d^3r = -MH_0(t) \tag{34}$$

where $m(r)$ is the magnetic moment of the cylinder at r and M is the total magnetization of the system. In the presence of the external magnetic field, $H_0(t) = H_0 e^{i\omega t}$, a magnetic moment $M_\omega e^{i\omega t}$ is established, and M_ω is given by

$$M_\omega = (\pi\mathscr{L}/c) \int_a^b r^2 J_\omega(r)\, dr \tag{35}$$

The generalized susceptibility $\alpha_M(\omega)$ of the magnetic moment is defined by

$$\alpha_M(\omega) = M_\omega/H_0$$

Hence,

$$\alpha_M(\omega) = -[b^2\mathscr{L}/4D(\omega)][I_2(qb)\, K_2(qa) - I_2(qa)\, K_2(qb)] \tag{36}$$

For superconducting cylinders, the variance of the magnetic moment is given by (12) and (36),

$$\begin{aligned} \operatorname{Var}(M) = (b^2\mathscr{L}/4)\{D_\lambda^{-1}[I_2(b/\lambda)\, K_2(a/\lambda) - I_2(a/\lambda)\, K_2(b/\lambda)]\} \\ - D_s^{-1}[I_2(b/\lambda_s)\, K_2(a/\lambda_s) - I_2(a/\lambda_s)\, K_2(b/\lambda_s)]\}\, k_B T \end{aligned} \tag{37}$$

where D_s is defined by (24) and D_λ by (29). For a normal metal, $\alpha_M(0) = 0$, and we have

$$\operatorname{Var}(M) = (b^2\mathscr{L}/4D_\lambda)[I_2(b/\lambda)\, K_2(a/\lambda) - I_2(a/\lambda)\, K_2(b/\lambda)]\, k_B T$$

In the case where the cylinder is not hollow,[6] we obtain, by putting $a = 0$,

$$\alpha_M(\omega) = -(b^2\mathscr{L}/4)\, I_2(qb)/I_0(qb)$$

The variance of the magnetic moment of the superconducting solid cylinder is given by

$$\operatorname{Var}(M) = \frac{b^2\mathscr{L}}{4}\left[\frac{I_2(b/\lambda)}{I_0(b/\lambda)} - \frac{I_2(b/\lambda_s)}{I_0(b/\lambda_s)}\right] k_B T$$

In the limit of a thin cylinder, i.e., $a \gg \lambda_s \gg d$ and $ad/2\lambda^2 \gg 1$, we have, near the critical temperature, where $\lambda_s \gg \lambda$,

$$\operatorname{Var}(M) = \frac{R^2\mathscr{L}}{4}\,\frac{k_B T}{1 + (Rd/2\lambda^2)}$$

From Eq. (31), I_ω can be written as

$$I_\omega = H_0(\mathscr{L}c/4\pi)\{[2/q^2a^2D(\omega)] - 1\}$$

Therefore,

$$\alpha_{IM}(\omega) = (c\mathscr{L}/4\pi)\{[2/q^2a^2D(\omega)] - 1\}$$

Using Eq. (12) with $A = M$ and $B = I$, we have

$$\mathrm{Cov}(M, I) = (\mathscr{L}ck_BT/2\pi a^2)[(\lambda_s^2/D_s) - \lambda^2/D_\lambda] \tag{38}$$

Equations (27) and (36); (30) and (37); and (32) and (38) are, in fact, not independent. Using (17)–(19) and the definition of magnetization (35), it is straightforward to show that

$$\Phi_{t\omega} = 2\pi bA_\omega(b) = \pi b^2H_0 + 4\pi(M/\mathscr{L}) \tag{39}$$

This is just another form of the well-known relation

$$B = H + 4\pi m$$

where B is magnetic flux density, H is magnetic field, and m is the magnetization per unit volume.

Using (39), it is easy to show that (36) can be derived from (27), (37) from (30), and (38) from (32).

4. SPECTRA AND RELAXATION OF MAGNETIC FLUX AND MAGNETIZATION

To evaluate the spectra, we are going to use the London two-fluid model,[18–21] which is valid for superconductors with localized electromagnetic response. From the two-fluid model, the current density and the vector potential are related by

$$\mathbf{J}_\omega(\mathbf{r}) = -Q(\omega)\,\mathbf{A}_\omega(\mathbf{r}) \tag{40}$$

where

$$\frac{4\pi}{c}\,Q(\omega) = \frac{1}{\lambda_s^2} + \frac{1}{\lambda_n^2}\,\frac{i\omega}{[i\omega + (1/\tau_n)]}$$

$$\lambda_s = (mc^2/4\pi n_s e^2)^{1/2}, \qquad \lambda_n = (mc^2/4\pi n_n e^2)^{1/2}$$

with n_n the electron density in the excited state, and τ_n is the momentum relaxation time of the normal electrons. We observe that, at $\omega = 0$ and $\omega = \infty$, we obtain (15) and (16), which have been derived without using the two-fluid model. The low-frequency ($\omega < k_BT$, c/a)[7] spectrum of the magnetic flux is given by (11) and (27),

$$\begin{aligned} S_{\Phi_t}(\omega) = {} & -(16\pi^2b/\mathscr{L})(k_BT/\omega) \\ & \times \mathrm{Im}\{[1/qD(\omega)][K_2(qa)\,I_1(qb) + I_2(qa)\,K_1(qb)]\} \end{aligned} \tag{41}$$

where $q^2 = (4\pi/c)\,Q(\omega)$. The autocorrelation function is given by

$$\begin{aligned} \mathrm{Cov}[\Phi_t(t), \Phi_t(0)] &= \langle[\Phi_t(t) - \bar{\Phi}_t][\Phi_t(0) - \bar{\Phi}_t]\rangle \\ &= \int_{-\infty}^{\infty} d\omega\,[e^{i\omega t}S_{\Phi_t}(\omega)/2\pi] \end{aligned}$$

[7] The boundary condition (19) requires $\omega < c/a$.

To go further, we have to consider some limiting cases. In the case of thin cylinder, $|q|d \gg 1$ and $|q|a \gg 1$, we have

$$S_{\Phi_t}(\omega) = -\frac{2ck_BTL}{\tau_n}\frac{Rd/2\lambda_n^2}{(1 + Rd/2\lambda^2)^2}\frac{1}{\omega^2 + (1/\tau^2)}$$

where L is the self-inductance, $L = 4\pi^2R^2/c\mathscr{L}$. The autocorrelation function is then given by

$$\mathrm{Cov}[\Phi_t(t), \Phi_t(0)] = ck_BTL\frac{Rd/2\lambda_n^2}{(1 + Rd/2\lambda_s^2)(1 + Rd/2\lambda^2)}e^{-|t|/\tau} \tag{42}$$

where

$$\tau = [(1 + Rd/2\lambda^2)/(1 + Rd/2\lambda_s^2)]\,\tau_n \tag{43}$$

Above the critical temperature, the relaxation time becomes

$$\tau_n' = [1 + (Rd/2\lambda^2)]\,\tau_n$$

This is the relaxation time for the magnetic flux in a normal metal cylinder. Usually, $Rd/2\lambda^2$ is much greater than one, and τ_n can then be written in the familiar form

$$\tau_n' = L/cR_n$$

where R_n is resistance and L is the self-inductance.

Consider a dirty film where τ_n is, say, of the order 10^{-13} sec. Then, $1/\tau_n$ is greater than k_BT. However, in most realistic cases, $Rd/2\lambda^2$ is much greater than one, and we can find a certain range of temperature near T_c such that

$$[1 + (Rd/2\lambda^2)]/[1 + (Rd/2\lambda_s^2)]\,\tau_n > 1/k_BT$$

is valid. For example, when $1 \gg k_BT\tau_n \gg 2\lambda^2/Rd$, the temperature range where the classical limit is valid is given by

$$\tfrac{1}{2}\tau_nk_BT > 1 - T/T_c$$

In the case of solid cylinder, the spectrum of the magnetic flux simplifies to

$$S_{\Phi_t}(\omega) = -(16\pi^2b/\mathscr{L})(k_BT/\omega)\,\mathrm{Im}\{I_1(qb)/qI_0(qb)\}$$

The peak value of $S_{\Phi_t}(\omega)$ at $\omega = 0$ is equal to

$$S_{\Phi_t}(0) = k_BT\frac{16\pi^2b}{\mathscr{L}}\frac{\tau_n\lambda_s^3}{\lambda_n^2}\left\{\frac{2I_1(b/\lambda_s)}{I_0(b/\lambda_s)} - \frac{b}{\lambda_s}\left[\left(\frac{I_1(b/\lambda_s)}{I_0(b/\lambda_s)}\right)^2 - 1\right]\right\}$$

At $T = T_c$, the peak value becomes

$${}_nS_{\Phi_t}(0) = k_BT_c(16\pi^2b/\mathscr{L})\,\tau_n\lambda(b/2\lambda)^3$$

The variation of $S_{\Phi_t}(\omega)/S_{\Phi_t}(0)$ with $\omega\tau_n$ is plotted in Fig. 3 for $b/\lambda = 100$, $T_c/T = 0.8$, 0.9, and 0.98. The relation $\lambda/\lambda_s = \{2[1 - (T/T_c)]\}^{1/2}$ has been used. From the line-

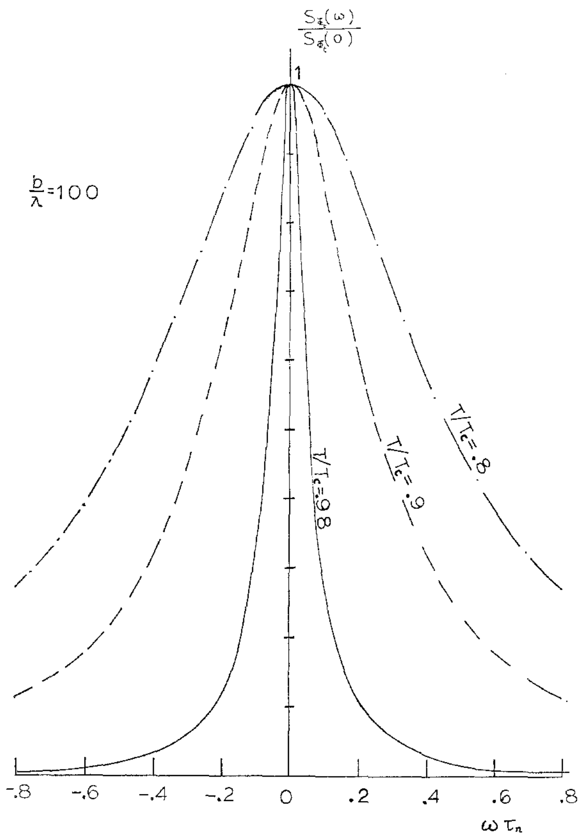

Fig. 3. Spectra of magnetic flux in a solid, superconducting cylinder.

shape of the spectrum, we notice that the relaxation time τ of Φ_t in a solid superconducting cylinder increases with temperature. When $T/T_c = 0.9$, τ is about an order higher than τ_n. When $T/T_c = 0.98$, τ is two order higher. The variation of $S_{\Phi_t}(0)/{}_nS_{\Phi_t}(0)$ with T/T_c is plotted in Fig. 4 for $b/\lambda = 100$.

Using (40), the Fourier transform of the superconducting component of the current response is

$$I_{s\omega} = (c/\lambda_s^2 q^2)\{[2/q^2a^2D(\omega)] - 1\}(\tilde{I}_0/c)$$

Using (12) with $A = \Phi_t$ and $B = I_s$, the covariance of the superconducting component of the current and the total magnetic flux is found to be

$$\mathrm{Cov}(I_s, \Phi_t) = \left[\frac{2}{a^2}\left(\frac{\lambda_s^2}{D_s} - \frac{\lambda^4}{\lambda_s^2 D_\lambda}\right) - \frac{\lambda^2}{\lambda_n^2}\right] ck_BT \tag{44}$$

Since $\mathrm{Cov}(I_n, \Phi_t)$ is equal to the difference between $\mathrm{Cov}(I, \Phi_t)$ and $\mathrm{Cov}(I_s, \Phi_t)$, we obtain from (32) and (44) an expression for the covariance between the normal component of the current and the total magnetic flux:

$$\mathrm{Cov}(I_n, \Phi_t) = (\lambda^2/\lambda_n^2)[1 - (2\lambda^2/a^2D_\lambda)]\, ck_BT$$

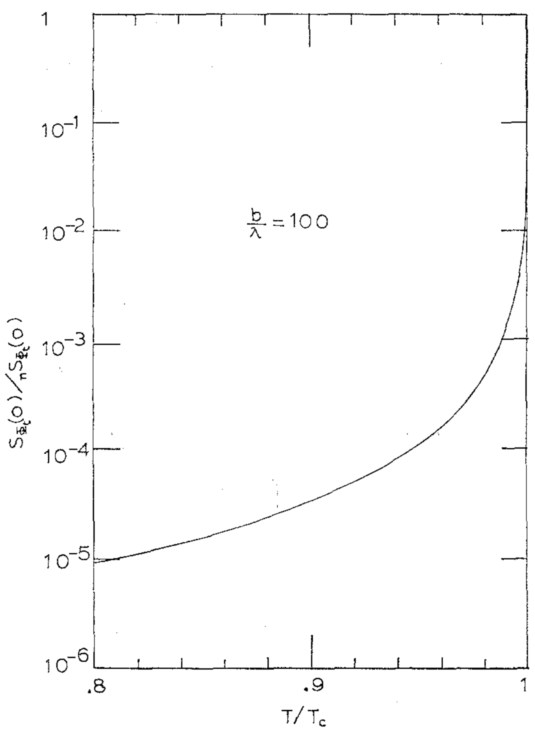

Fig. 4. Variation of the spectrum of magnetic flux at zero frequency with temperature in a solid, superconducting cylinder.

Similarly, the spectrum for magnetization is given by (11) and (36):

$$S_M(\omega) = k_B T(b^2 \mathscr{L}/2) \operatorname{Im}\{[1/D(\omega)][I_2(qb) K_2(qa) - I_2(qa) K_2(qb)]\} \tag{45}$$

For a thin cylinder, the autocorrelation function is

$$\operatorname{Cov}[M(t), M] = k_B T \frac{R^2 \mathscr{L}}{4} \frac{Rd/2\lambda_n^2}{(1 + Rd/2\lambda^2)(1 + Rd/2\lambda_s^2)} e^{-t/\tau} \tag{46}$$

where τ is defined by (43). Equations (41) and (45), (42) and (46) are connected by (39).

APPENDIX

We can derive an expression for $Q(\omega)$ at $\omega = \infty$ from the perturbation theory by following very closely the arguments of Martin and Schwinger.[22] From perturbation theory, the current density is given by

$$\langle j_\mu(\mathbf{r}, t)\rangle = \sum_\nu \int_{-\infty}^{\infty} K(\mathbf{r}, t; \mathbf{r}', t')\, A_\nu(\mathbf{r}', t')\, d^3r'\, dt' \tag{A1}$$

where the electromagnetic response kernel is defined by

$$K_{\mu\nu}(\mathbf{r}t;\, \mathbf{r}'t') = (i/c)\, \theta(t - t')\langle[j_\mu(\mathbf{r}, t), j_\nu(\mathbf{r}', t')]\rangle - \delta_{\mu\nu}\, \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t')(ne^2/mc) \tag{A2}$$

$\theta(t)$ is a unit step function defined by (8); n is the electron density.

As we are only interested in superconductors with localized electromagnetic response, we can write

$$\langle[j_\mu(\mathbf{r}, t), j_\nu(\mathbf{r}', t')]_-\rangle = \delta_{\mu\nu}\, \delta(\mathbf{r} - \mathbf{r}')\, \eta(t - t') \tag{A3}$$

The factor $\delta_{\mu\nu}$ is due to isotropy of the medium. Equation (A3) can be written as

$$\langle[j_\mu(\mathbf{r}, t), j_\nu(\mathbf{r}', t')]_-\rangle = (\partial/\partial t)\, i\, \delta_{\mu\nu}\, \delta(\mathbf{r} - \mathbf{r}') \times \int_{-\infty}^{\infty} d\omega\, \{[\exp i\omega(t - t')]\, \sigma(\omega)/\pi\} \tag{A4}$$

where

$$\omega\sigma(\omega) \equiv -\tfrac{1}{2} \int_{-\infty}^{\infty} dt\, [e^{-i\omega t}\eta(t)] \tag{A5}$$

Due to conservation of charge, we have

$$-i\nabla\delta(\mathbf{r} - \mathbf{r}') \int_{-\infty}^{\infty} d\omega\, \{[\exp i\omega(t - t')]\, \sigma(\omega)/\pi\} = e\langle[\rho(r, t), \mathbf{j}(\mathbf{r}, t)]_-\rangle \tag{A6}$$

where $\rho(r, t)$ is the density operator.

Applying the commutation rule to the right-hand side of (A6), we obtain the sum rule by comparing both sides:

$$\int_{-\infty}^{\infty} d\omega\, [\sigma(\omega)/\pi] = ne^2/m \tag{A7}$$

where $n = \langle\rho\rangle$ is the electron density of the metal. With the aid of (A4) and the sum rule (A7), the kernel (A2) can be written as

$$K_{\mu\nu}(\mathbf{r}, t; \mathbf{r}', t') = -\delta_{\mu\nu}\delta(\mathbf{r} - \mathbf{r}')\, \frac{1}{c}\frac{\partial}{\partial t} \left\{\theta(t - t') \int_{-\infty}^{\infty} \frac{d\omega}{\pi}\, [\sigma(\omega) \exp i\omega(t - t')]\right\} \tag{A8}$$

Substituting (A8) into (A1) we obtain the Fourier transform of $\langle\mathbf{j}(\mathbf{r}, t)\rangle$:

$$\mathbf{j}_\omega(\mathbf{r}) = -Q(\omega)\, \mathbf{A}_\omega(\mathbf{r})$$

where $Q(\omega)$ is given by

$$Q(\omega) = (i\omega/c) \int_{-\infty}^{\infty} dt\, [e^{-i\omega t}\theta(t)] \int_{-\infty}^{\infty} d\bar{\omega}\, [e^{i\bar{\omega} t}\sigma(\bar{\omega})/\pi]$$

After some straightforward computation,

$$Q(\omega) = \frac{\omega}{c} \int_{-\infty}^{\infty} \frac{d\bar{\omega}}{\pi} \frac{\sigma(\bar{\omega})}{\omega - \bar{\omega}} + \frac{i\omega}{c} \sigma(\omega)$$

As $\mathrm{Im}\{Q(\omega)\}$ is an odd function of ω, it vanishes at $\omega = \infty$. Therefore, $\sigma(\omega)$ goes to zero faster than ω^{-1} as $\omega \to \infty$. Hence, as $\omega \to \infty$, we have[23]

$$\lim_{\omega \to \infty} Q(\omega) = \lim_{\omega \to \infty} \mathrm{Re}\{Q(\omega)\} = \int_{-\infty}^{\infty} d\bar{\omega}\, [\sigma(\bar{\omega})/\pi c] = ne^2/mc$$

or $(4\pi/c)\, Q(\infty) = 1/\lambda^2$, which is Eq. (16). In the derivation of $\lim_{\omega\to\infty} Q(\omega)$, we need not make any distinction between superconductor and normal metal. The above arguments also apply to thin films.

In Sections 3 and 4, we have simply denoted $\langle \mathbf{j}(\mathbf{r}, t) \rangle$ by $\mathbf{j}(\mathbf{r}, t)$.

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